In Search of Newton
Combined Calculus and Physics Materials
for Studio Calculus/Physics

University of New Hampshire

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and all the students who took this course!

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Contents

I Introduction 5
  1 Overview of Course and Materials 7

II Kinematics 9
  2 Emily’s Walk 11
  3 Emily’s walk - Redux 15
  4 Kinematics Problem Solving 19

III Dynamics 21
  5 Newton’s Method 23
  6 Nasty Canasty vs. Monty Gue 27
  7 Drag Force on a Coffee Filter 31

IV Conservation Laws 39
  8 Balance Point 41
  9 Air Drag and Euler’s method 49

V Rotational Motion 53
  10 Moment of inertia 55

VI Oscillations and Waves 61
  11 Description of Motion for Object on a Spring 63
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>Forces for Object on a Spring</td>
<td>71</td>
</tr>
<tr>
<td>13</td>
<td>Fourier Series</td>
<td>77</td>
</tr>
<tr>
<td>14</td>
<td>Problem Solving: Damped motion</td>
<td>83</td>
</tr>
<tr>
<td>VII</td>
<td>Electrostatics</td>
<td>85</td>
</tr>
<tr>
<td>15</td>
<td>How fast is the comet going?</td>
<td>87</td>
</tr>
<tr>
<td>16</td>
<td>When will the comet hit?</td>
<td>89</td>
</tr>
<tr>
<td>17</td>
<td>$\vec{E}$ field due to a bar of charge</td>
<td>95</td>
</tr>
<tr>
<td>18</td>
<td>Calculating $V$ due to a Charge Distribution</td>
<td>103</td>
</tr>
<tr>
<td>19</td>
<td>Neon Lights</td>
<td>107</td>
</tr>
<tr>
<td>VIII</td>
<td>Circuits</td>
<td>109</td>
</tr>
<tr>
<td>20</td>
<td>Discharging an RC Circuit</td>
<td>111</td>
</tr>
<tr>
<td>21</td>
<td>Charging an RC Circuit</td>
<td>117</td>
</tr>
<tr>
<td>22</td>
<td>Design of an RC Circuit</td>
<td>123</td>
</tr>
<tr>
<td>IX</td>
<td>Magnetism</td>
<td>127</td>
</tr>
<tr>
<td>23</td>
<td>A new circuit element</td>
<td>129</td>
</tr>
<tr>
<td>24</td>
<td>Martian Gas Gauge</td>
<td>131</td>
</tr>
<tr>
<td></td>
<td>Or how I came to love series and differential equations</td>
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</tbody>
</table>
Part I

Introduction
Activity 1

Overview of Course and Materials

This is a collection of activities designed over three years for a combined freshman level calculus/physics course at the University of New Hampshire. The development of this course was funded by the National Science Foundation Division of Undergraduate Education (NSF-DUE-9752485).

What does this course look like? This is really two courses (eight credits) that meets for 10 hours a week (two 2-hour sessions of physics, two 2-hour sessions of calculus, one combined 2-hour session). The topics covered are essentially the same found in a standard physics or calculus course (though in the physics we have cut back a bit on breadth in favor of depth); we use standard textbooks in each course. However, the calculus topics have been reordered so that students have the mathematics they need to do the physics. For example, students can take antiderivatives of polynomials by the end of the first month of class so that in physics students can work with the idea that velocity is the antiderivative of acceleration instead of focusing only on constant acceleration equations. Also, during class time students are working in groups a good deal of the time; class is not all lecture. The two-hour class session allows us to intermingle lab activities as appropriate (there is no separate lab time). This is based on the model of studio physics pioneered at Rensselaer Polytechnic Institute.

What is in this booklet? The activities in this booklet are those that combine to a large degree both the mathematics and the science; they do not in any sense make a complete curriculum. We have collected them in one place for instructors who might want to supplement their current curriculum with these activities.

Other booklets from this same course include the instructor manual, the calculus activities, the calculus lectures, the physics activities, suggestions for physics lectures, and a collection of calculus/physics projects.
Part II

Kinematics
Activity 2

Emily’s Walk

During our last class we worked with displacement \( \Delta \vec{x} \equiv \vec{x}_2 - \vec{x}_1 \) and found that it was a vector. On this worksheet we will work with displacement, velocity and acceleration and learn about their vector properties.

**Constant speed** Emily walks 6 meters east and then 6 meters north-east. She walks at a constant speed of 2 meters per second for the whole trip and begins walking at \( t = 0 \) seconds.

1. In lecture we will discuss what is her final position (that is, what is her displacement) after the whole trip? Draw a picture using vectors and state the result in both component form (with \( \hat{i} \) and \( \hat{j} \) notation) and as magnitude and direction.

2. **First leg of the trip**
   
   (a) What is her eastward velocity on the first leg of the trip?
   (b) What is her northward velocity on the first leg of the trip?
(c) Express her velocity on this leg of the trip in \( \hat{i} \) and \( \hat{j} \) notation.

(d) Use your three previous answers and anti-derivatives to write down her position \textbf{in vector notation} for her first leg of the trip.

(e) What is the domain of this function? What is the range? How does it relate to your sketch?

3. Second leg of the trip

(a) What is her eastward velocity on the second leg of the trip?

(b) What is her northward velocity on the second leg of the trip?
(c) Express her velocity on this leg of the trip in $\hat{i}$ and $\hat{j}$ notation.

(d) Use antiderivatives to find her position in vector notation for the second leg of the trip. Be very careful as you evaluate the constants.
4. Whole trip

(a) Calculate her northward and eastward average velocities for the whole trip.

(b) Express her average velocity for the whole trip in $\hat{i}$ and $\hat{j}$ notation.

(c) Find the magnitude and direction of her average velocity for the whole trip.

(d) How does the direction of the average velocity compare with the direction of her total displacement? Are they the same or different?

(e) Write down the equation that relates average velocity to total displacement. Discuss with your group the answer to your last question in light of the definition.
Activity 3

Emily’s walk - Redux

Non-constant speed This is an exercise in working with displacement, velocity and acceleration when speeds are not constant. Emily decides to get a bit more tricky on you. She goes for a walk, and her coordinates are changing in time:

\[
x(t) = 2t \text{ m/s} - t^2 \text{ m/s}^2, \\
y(t) = t^2 \text{ m/s}^2.
\]

1. Verify that the following graph is the path that she takes by calculating her location at \( t = 0, 1, 2 \) and 3 second, plotting these locations as large circles and verifying that they lie on the curve.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 sec</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 sec</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In all of the following, give your answers in vector notation, that is, using \( \hat{i} \) and \( \hat{j} \).

2. What is the average rate of change of her position from \( t = 1 \) to \( t = 2 \)?

3. What is the average rate of change of her position from \( t = 1 \) to \( t = 1.5 \)?

4. What is the average rate of change of her position from \( t = 1 \) to \( t = 1.25 \)?

5. You have just calculated the first three terms in a sequence. Does the sequence appear to be converging? If so, to what vector?

6. What is her velocity vector at any given time?
7. Calculate her velocity vector at $t = 0, 1, 2$ and $3$ seconds.

8. Sketch the velocity vectors on the plot. In order to do this, you must pick a scale: 1 meter on the plot is equal to how many meters/second?

9. What is her acceleration vector as a function of time?

10. Calculate her acceleration vectors at $t = 0, 1, 2$ and $3$ seconds.

11. Sketch her acceleration vectors on the plot. What scale will you use?
Activity 4

Kinematics Problem Solving

1. You are helping to design a machine that will accelerate electrons using charged plates at the end of long hollow tubes. As the electrons move between the plates, the charge on the plates increases, so the acceleration of the electrons increases with time as follows: \( a(t) = 5 \times 10^6 t^3 \). If the electron starts with an initial velocity of zero, how long does it take for the electron to travel .5 meters between the plates? (You look up information about electrons and find that they have a mass of \( 9.11 \times 10^{-31} \) kg and a charge of \( 1.6 \times 10^{-19} \) Coulombs.)

\[
\begin{align*}
\text{v=0 at left edge} \\
\end{align*}
\]

2. (Note: this situation is a bit contrived, but the solution is still instructive!) There is a bottomless pit in the basement of the physics building (in the Special Equipment room). Inside this specially constructed pit (lined with kryptonite). This kryptonite changes the acceleration due to gravity from the usual \( 9.8 \) m/s\(^2\) downward to a highly surprising \( 2t \) m/s\(^2\) upward, where \( t = 0 \) is the instant something is thrown into the pit. You toss an apple downward into the pit at \( t = 0 \)s with a velocity of 4m/s. Take the initial position to be zero. Considering this to be a one-dimensional problem, find expression for the velocity and position of the apple at any given time (as long as it remains in the pit). Does the apple ever come back to you or does it keep going?

3. You have a toy rocket that you shoot up in the air. For the first four seconds while the rockets are firing, they provide an acceleration of 21 m/s\(^2\). After four seconds the rocket cuts out and only the acceleration due to gravity is present. How high will the rocket go and how long will it take to reach to top of its motion?
Part III
Dynamics
1. Sketch the tangent line to the curve at $t = 3.5$.

2. Approximate the time when the tangent line crosses the $t$-axis.

3. Sketch the tangent line to the curve for this new time value.

4. Approximate the time when this new tangent line crosses the $t$-axis.

5. Sketch the tangent line to the curve for this new time value.

6. Approximate the time when this new tangent line crosses the $t$-axis.
ACTIVITY 5. NEWTON’S METHOD

7. Sketch the tangent line to the curve for this new time value.

8. Approximate the time when this new tangent line crosses the $t$-axis.

9. What value of time do you approach? What is special about this time value?
Find the general formula:

1. Given $t_0$, $f(t_0)$, and $f'(t_0)$, find the formula for the tangent line using the slope-intercept form for a line.

2. Find the value of $t$ at which the tangent line crosses the $t$-axis. This equation is the basis of Newton’s method.

3. Now we want you to find the root of $f(t) = t^3 - 1.4t^2 -.9t - 3.6$. We give you an initial guess of $t_0 = 3.5$. You will find that Matlab is a great help here. Below is some sample code to get you started. Use the up arrow to recall previous equations so you need not retype the formulas for each iteration.

   ```
   t0=3.5
   f=t0^3-1.4*t0^2-.9*t0-3.6
   fp=[put your derivative here]
   t=t0-f/fp
   t0=t
   ```
4. Fill in the following table for $f(t) = t^3 - 1.4t^2 - .9t - 3.6$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$t_n$</th>
<th>$f(t_n)$</th>
<th>$f'(t_n)$</th>
<th>$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
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<td>2</td>
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</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Activity 6

Nasty Canasty vs. Monty Gue

Archvillian Nasty Canasty is in his hand-car moving at 50 metres per second eastward when he notices that his nemesis, Monty Gue, is traveling straight at him, moving westward, on the same set of tracks. Monty is in a bullet-proof train and is moving at a constant speed of 50 metres per second. Nasty Canasty orders his slow moving henchmen to reverse the direction. As they do so, they apply an acceleration of \( t/2 \, \text{m/s}^3 \) westward. The acceleration is applied when Monty is only 1000 meters away from Nasty. Will Nasty escape?

1. Gather information - remember, do not do any calculations yet!

   • What information are you given?

   • Draw a picture of the situation and label your coordinate system.

   • Qualitatively describe and sketch the motion of both vehicles. Based on these sketches, is it possible that Nasty escapes? Is it possible that he doesn’t?
ACTIVITY 6. NASTY CANASTY VS. MONTY GUE

- Does there appear to be any missing information? Any extraneous information?

- What is the question asking? That is, what equation is the question asking us to solve?

2. Organize - don’t do any calculations yet!

- What general approach will you use to solve this (e.g. Fnet=ma? Estimating $x$ from the plot of $a$? something else?) To decide which approach is best, find the method connects what you know with what you want to find out.
3. Use the approach on the last page to find the answer to the question.

- Setup the equation(s) you need to solve.

- You should now see that you need to use Newton’s method to solve this problem. What equation do you want to find the root of?

- What is the derivative of that equation?
• In order to find the root, what is a good guess for \( t_0 \)? Hint: When would they hit if Nasty didn’t accelerate? Will hitting occur before or after this?

• Use Matlab to find the root of this equation.
  – After the first iteration, be sure to use the up arrow key so you don’t have to retype the same commands each iteration.
  – Monitor the value of the function we are finding the roots of. Is it taking on reasonable values?
  – When is your answer close enough? Hint: think about significant digits!
  – Write down the numbers below for each iteration:

<table>
<thead>
<tr>
<th></th>
<th>( t_n )</th>
<th>( f(t_n) )</th>
<th>( f'(t_n) )</th>
<th>( t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
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<tr>
<td>5</td>
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</tr>
</tbody>
</table>

4. Check.

• Are your units on the answer correct?

• Does your answer seem reasonable?

• Can you solve it another way to check?

• Why did we ask this question?
Activity 7

Drag Force on a Coffee Filter

1. If you consider gas molecules in the air to be small hard spheres, how would you explain or describe the force known as drag force or air resistance?

2. Consider a coffee filter falling. What features of the physical situation should affect the magnitude of the force? Are they directly or indirectly proportional to the force? Explain! (Many of you may be familiar with the formula for air resistance, but please take a few minutes to try and make sense of all those terms that are there!) (Don’t spend any more than 5 minutes considering these options, just get a feel for what you think is happening.)

width of filter

height of filter

shape of filter

velocity of filter

mass of filter

material filter is made of

density of air

other?
3. In fact the book tells us that

$$F_{\text{drag}} = \frac{1}{2} C \rho A v^p$$

where $C$ is the drag coefficient, $A$ is the cross-sectional area of the falling object, $\rho$ is the density of air, and $p$ is a power that we need to determine. The book claims that $p = 2$, we will verify that value. (When we learn about work and energy, we will be able to prove this formula, if we make one reasonable assumption.)

Reconcile any major differences between your predictions on the last page and the formula from the book.

4. Sketch the free body diagram of the filter just after it begins to fall. Be sure to include the acceleration and velocity vectors in the diagram.

5. Write an equation which gives the acceleration in terms of the forces. Be very careful that the signs are correct. Take the downward direction to be positive.

6. Using your last answer, write down an expression for $\dot{v}$ in terms of $C, \rho, A, v, g$ and $m$. 
7. From the above differential equation, does $dv/dt$ increase or decrease as the velocity increases from zero? Explain.

8. Can $dv/dt$ ever be zero? Describe the motion when $dv/dt = 0$.

9. Find an expression for $v_{\text{terminal}}$ (the velocity when $dv/dt = 0$) in terms of $C, \rho, A, g$ and $m$.

1. Now we that we have the general equations, we will look at a particular example of the drag force. On the next page is the slope field where we picked the constants so that

$$\frac{1}{2} \frac{C \rho A}{m} = 2$$

and $p = 2$. Recall that the short lines give the slope of $v$ at each point. What is $v_{\text{terminal}}$ for this value of the parameters?

2. What is the value of the slope of $v(t)$ when $v = v_{\text{terminal}}$?

3. Does this agree with the slope field?
4. In general, does the value of the slope depend on \( v \)? does this agree with the slope field below?

5. In general, does the value of the slope depend on \( t \)? does this agree with the slope field below?

6. State in words how you would sketch an approximate solution to this differential equation if \( v(0) = 0 \) m/s. Does your solution have to hit a tangent line all the time?

7. Sketch the approximate solutions for \( v(0) = 0 \) m/s and \( v(0) = 3 \) m/s.
**Experiment** We will now experimentally determine the power $p$.

Given the formulas you have just written, why can we not determine the power from finding the terminal velocity for just one filter? Hint: what values do we know, what values are unknown?

We can, however, find the power $p$ by finding the terminal velocity for several different runs, each with a different number of filters (from one to five) and therefore a different mass. We will find the power by fitting terminal velocity vs. mass to a power law form.

1. Turn on the computer and ULI. Open up MacMotion program.
2. Make sure that the UMD is pointing straight down by making sure it tracks your hands if they are directly underneath the UMD.
3. Repeat the following procedure five times, beginning with one filter, and going up to five filters:
   
   (a) If you have more than one filter, attach the filters to each other with a small piece of masking tape.
   
   (b) Find the mass of all the filters using the electronic balance and write it in the table below.
   
   (c) Drop the filter as close as possible to the UMD. The initial data will be incorrect because we’re too close to the detector, but this will allow the filter to reach terminal velocity before it hits the floor.
   
   (d) Practice dropping the filter so that it goes down straight without wobbling. **This step is the most important to getting good data!** You may have to take several runs to get good data. Do not include the data for five filters if it looks as though terminal velocity was not reached. How can you tell if terminal velocity was reached?

   (e) Once you have good data, find the value of terminal velocity by highlighting the time interval during which $v$ was nearly constant. Use “statistics” (under “analyze” menu) at this stage. Enter this data in the table below. (Note that the data for zero filters you can fill in without experiment.) Estimate the error on your terminal velocity. Explain your procedure for estimation.

   (f) You can print out one good graph of $v(t)$ for the three or four filter run.
<table>
<thead>
<tr>
<th>number of filters</th>
<th>mass (gm)</th>
<th>terminal velocity (m/s)</th>
<th>error in velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>5</td>
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<td></td>
</tr>
</tbody>
</table>

**Data Analysis**

1. Open up the program Graphical Analysis on the computer.

2. Enter the values of the masses under X data, the velocity under Y data.

3. Now, let’s think about where we’re headed with the program. Below, write the formula for the terminal velocity in terms of $C, \rho, A, v, g$ and $m$.

4. You will be fitting your data to a power law $y = Ax^B$. Relate each of the general variables in the fitting formula $(y, x, A, B)$ to the variables in the original equation $(v_t, C, \rho, p, A, m, g)$:

   - $y =$
   - $x =$
   - $A =$
   - $B =$

5. From the answer to your last question, how will fitting the data to a power law help you to determine $p$?
6. Let’s proceed with the fitting. In order to have a clear graph, do the following:

   (a) By clicking on Label and Unit portions of the table, give each set of data the correct label and units.

   (b) Under Graph, choose Rename to give your plot a meaningful name.

7. Under Graph and Graph Options... (last item) enter a reasonable error for all the velocity data points. (We would like to enter a different error for each data point, but the program cannot do that.) Be sure to uncheck the percentage error button. The under Graph, select Error Bars so that the error bars show.

8. Under Analysis choose Automatic Curve Fit and then Power - the program will find the best power law fit for the data. Print out the graph out if you’d like.

Conclusions

1. Based on your data and the fit by graphical analysis, what is the most reasonable value for $p$? How sure are you of your value of $p$? Is this in agreement with the text? (Note that you can choose ”manual data fit” under the ”analyze” menu and manually change the value of $B$ to see what other values seem to fit the data.)

2. Based on your value of $A$ the parameter from the fit, what is the value of $C=$ drag coefficient? Note that your answer should be around one. (You will have to estimate the cross-sectional area of the filter.)
Part IV

Conservation Laws
Activity 8

Balance Point

In the following set of exercises we will learn how to find the balance point of an object. The balance point is the place where you can place your finger and be able to support the entire object. We will see in the next class that this point is also essential for understanding motion of the object.

1. Finding the Balance Point Experimentally

Obtain a meter stick, two weight hangers and a set of weights. Place the weight hangers at the specified locations (measured from the center of the meter stick) and find the mass \( m_1 \) needed to make the center of the stick (here taken to be 0 cm) the balance point of the system. Note that the hangers themselves have a mass of about 20 kg; this must be included in \( m_1 \) and \( m_2 \). Finally, the smallest object we have has a mass of 10gm, so you will not be able to get the masses correct to better than 10 gm.

<table>
<thead>
<tr>
<th>( x_1 )</th>
<th>( m_1 )</th>
<th>( x_2 )</th>
<th>( m_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20 cm</td>
<td>20 cm</td>
<td>70 gm</td>
<td></td>
</tr>
<tr>
<td>-20 cm</td>
<td>40 cm</td>
<td>20 gm</td>
<td></td>
</tr>
<tr>
<td>-20 cm</td>
<td>15 cm</td>
<td>40 gm</td>
<td></td>
</tr>
</tbody>
</table>

2. Verifying the formula for the balance point

The book gives the following definition for finding the balance point:

\[
x_{\text{balance point}} \equiv \frac{1}{M_{\text{total}}} \sum_{i=1}^{N} m_i x_i
\]

where \( N \) is the total number of objects that we are considering.

In the following space verify that this formula gives the center of mass at the center of meter stick for each of the three situations given above. (Ignore the mass of the meter stick itself in this calculation.)
3. No-calculation balance points

In following three cases, imagine that the object is a thin sheet of metal of the specified shape and uniform density, and that you are trying to balance it with your finger while the object is horizontal.

Mark the balance point of a meter stick with no extra hanging weights.

Mark the balance point of a square book.

Mark the balance point of a circle.

How did you determine these balance points?
4. Calculating Balance points in two dimensions

Now imagine that you have a square plate on which you place three objects with the following locations and masses:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0m</td>
<td>0m</td>
<td>3 kg</td>
</tr>
<tr>
<td>1m</td>
<td>2m</td>
<td>4 kg</td>
</tr>
<tr>
<td>2m</td>
<td>1m</td>
<td>8 kg</td>
</tr>
</tbody>
</table>

Sketch the location of the objects on the coordinate system of the plate and use your intuition to guess the location of the balance point. Mark your guess on the sketch with a "g". (Ignore the mass of the plate itself in this calculation.)

Using the above formula for \( x \) balance point and

\[
y_{\text{balance point}} = \frac{1}{M_{\text{total}}} \sum_{i=1}^{N} m_i y_i
\]

calculate the balance point and mark that on your sketch with an 'x'.
5. Calculating the Balance point in pieces

To illustrate a useful technique we will calculate the balance point in a different way. First calculate the balance point of the first two objects only in the last section:

Now, consider those first two objects to be one object with a mass of 7 kg located at their balance point. Now calculate the balance point for the "7 kg object" and the 8 kg object.

You should find that the balance point calculated in this section and the last are the same. If not, go back and check your calculations.

For those of you that like proofs, can you show that these balance points must be the same?

6. Calculating Balance points for complex objects

Use the ideas developed in this sheet to calculate the balance point for the following situation. Be sure to indicate what you are doing and why. The mass of the rectangle is 5kg, the mass of the circle is 3kg.
7. Calculating the Balance point for even more complex objects

Imagine that you have a rectangular sheet of length \( l = 1 \) m and width \( w = 1/7 \) m constructed so that the density is

\[
\text{density} = 3x^2 = \frac{\text{mass}}{\text{area}}
\]

Where \( x \) is the distance along the sheet, with \( x = 0 \) at the left side and 3 is in units of kg/m\(^4\). With this density, the sheet is much heavier on the right end than on the left. Sketch the density vs. \( x \) below. Calculate the values at each end of the bar.

Mark with a "g" where you would guess the balance point is located.

What is the value of \( y_{\text{balance point}} \)? Explain.

Calculating \( x_{\text{balance point}} \) is more difficult and will take several steps. Why do none of our other methods that we have used so far work here?

In this case we will need to approximate the total mass (\( \sum m_i \)) and first moment of the mass (defined as \( \sum m_i x_i \), the numerator in the center of mass formula) by breaking the sheet into pieces and approximating the density of each piece to be constant. (This is consistent with what we did in the last exercise: calculate the center of mass in pieces.) Do you expect this approximation to be accurate for two pieces? four pieces? one thousand pieces? Explain.
(a) To calculate $x_{\text{balance point}}$, first imagine that the mass of the sheet is located at only two points as illustrated below.

Fill out the chart below, keeping numbers as fractions, keep the powers explicit (e.g. you should have terms like $(1/4)^2$, not $1^2/8$). The reason for this is that we are looking for a general pattern which will not be obvious otherwise.

$$
\begin{array}{|c|c|}
\hline
\Delta x & \text{piece 1} & \text{piece 2} \\
\hline
\text{size of piece} & & \\
\hline
x & & \\
\hline
\text{location of dot} & & \\
\hline
\text{density at the dot} & & \\
\hline
\text{area of the pieces} & & \\
\hline
\text{mass assuming constant density} & & \\
\hline
mx & & \\
\hline
\text{first moment of the mass} & & \\
\hline
\end{array}
$$

(b) Do the same thing as in the previous question, considering that all the mass is located at four evenly spaced points along the sheet.
(c) We have calculated the terms for the total mass $\sum m_i$ and first moment $\sum m_i x_i$. Rewrite the terms in this sum using the density instead of the mass.

(d) Based on your answers to the last two questions, come up with a general form for the terms of the $i$'th piece if you had $n$ points.

<table>
<thead>
<tr>
<th>piece $i$ ($0 \leq i \leq n$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta x$</td>
</tr>
<tr>
<td>$x$</td>
</tr>
<tr>
<td>density</td>
</tr>
<tr>
<td>area</td>
</tr>
<tr>
<td>mass</td>
</tr>
<tr>
<td>$mx$</td>
</tr>
</tbody>
</table>

(e) Next you will use Matlab to calculate $x_{\text{balance point}}$ for several different values of $n$. Note that you can either take the points to be at the left edge ($0 \leq i \leq (n-1)$) or at the right edge ($1 \leq i \leq n$). Don’t forget to use the up arrow key to re-run the code for different values of $n$. Also, recall that `sum(f(2:N+1))` sums the second through N+1 elements in the vector $f$.

Write out your matlab code below:

Write down the values obtained below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>total mass (left)</th>
<th>total mass (right)</th>
<th>$mx$ (left)</th>
<th>$mx$ (right)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(f) Why do the values of total mass and balance point change as you change the number of points? Why do they change less as the number of points gets larger?
8. Balance point of the bird

An unusual bird will be passed around the class. Where do you think the balance point of the bird is? Hint: what other object behaves like the bird? Where is the center of mass of that object in relation to its support?

Why does the bird behave as it does?

9. Balance point of the bottle

Sketch the wine bottle and holder below. Where is the balance point? You can estimate the balance point from geometry and from looking at the wooden stand. Do those values agree?

10. Estimate of balance point

Below is the sketch of another bird made out of a piece of wood of uniform density. Which of the numbered points is most likely to be the balance point? Hint: first eliminate those that are obviously incorrect. For the rest, draw axes through the points, compare mass distribution above and below and left and right.
Activity 9

Air Drag and Euler’s method

Overview: In this worksheet you will use Euler’s method to find the solution to the differential equation for velocity for a falling coffee filter. You will compare your approximate solution with the exact solution, which we have seen, but don’t yet know how to derive.

Deriving the differential equation: Consider again an falling object subject to both gravitational force downward and air drag upward. You know from Newton’s laws that

\[ m\, a = -F_{\text{drag}} + mg \]

where we’ve taken the downward direction to be negative. From the text book,

\[ F_{\text{drag}} = \frac{1}{2} C \rho A v^2 \]

where \( C \) depends on the shape, (take \( C = 1 \)), \( \rho \) is the density of air, and \( A \) is the cross-sectional area of the object.

From the statements above, show that

\[ \dot{v} = -dv^2 + g \]

where \( d = \frac{1}{2} C \rho A / m \).
Understanding the Exact Solution: Next semester we will derive the solution to this differential equation (with $v(0) = 0$ m/s) is as follows:

$$v(t) = \frac{\sqrt{g}(-1 + e^{2\sqrt{gd} t})}{\sqrt{d}(1 + e^{2\sqrt{gd} t})}$$

For now, we will have to accept it as true. But we can at least check that it is reasonable:

- Given this formula, what is $v(0)$?
- What value does $v(t)$ approach when $t$ is large? Hint: Are the exponential terms large or small (compared to one) as $t$ gets very large?
- What is the terminal velocity (when $\dot{v} = 0$)? How does that compare with your last answer?

Sketch this solution on Matlab for $d = 2$ for the time interval $0 \leq t \leq .8$ seconds. (Yes, Matlab! - we will need the "for" loop later, so we might as well fire it up!) Sketch it here.

Does the plot of $v(t)$ look reasonable? Explain.
Estimate the solution using Euler’s method. Now use Euler’s method to solve this equation. Recall that the general form of Euler’s method is, given that

\[ y'(t) = f(x, y) \]

then

\[ y_n = y_{n-1} + \Delta x f(x_{n-1}, y_{n-1}) \]

- Write out Euler’s method for this problem. That is, what variable plays the role of \( y \)? \( x \)? what is \( f(x, y) \)?

- What is the initial value, \( y_0 \)?

- Write the Matlab code here:
• Use Euler’s method to solve this differential equation for $0 \leq t \leq .8$ seconds, using 10 steps. Plot the results on top of the exact solution by using the command `plot(t,v,'*'),t,vexact);` this way the approximate solution is show as "*", and the exact solution is shown as a line. Does your approximate solution seem reasonable? Explain.

• Use Euler’s method to solve this differential equation for $0 \leq t \leq .8$ seconds, using 20 steps. Plot your results as before. How does this solution compare to the previous one?

• Use Euler’s method to solve this differential equation for $0 \leq t \leq .8$ seconds, using 100 steps. Plot your results as before. How does this solution compare to the previous one?
Part V

Rotational Motion
Activity 10

Moment of inertia

First, recall that we found in the last activity that mass and the distance of that mass from the pivot both effect how difficult an object is to rotate. Moment of inertia is therefore the rotational analog to mass.

In this exercise we will calculate the moment of inertia for several objects, both point particles and extended objects. The moment of inertia is defined as

\[ I_A \equiv \sum_{\text{over all particles}} m_i r_{Ai}^2 \]

is the moment of inertia of a set of particles about axis \( A \), and \( r_{Ai} \) is the distance between the axis \( A \) and the \( i \)th particle. In this worksheet we will use this formula to calculate the moment of inertia of several different objects.

1. **Point particles** We begin with the easiest situation. Consider a system with two point particles: one of mass 3 kg at \( x = 0 \) cm, and another of mass 5 kg at \( x = 2 \) cm.

   • What is the moment of inertia of this system of particles for an axis through the origin?

   • for an axis through \( x = 2 \) cm?

   • for an axis through the center of mass?

   • for an axis through \( x = 100 \) cm?
2. **Intuition** Given your answers above, which do you suppose would have a greater moment of inertia through its center: a sphere of mass $M$ and radius $R$ or a hoop of the same mass and radius? (A hoop has essentially all of its mass at radius $R$) Explain.

3. **Center of Mass** One very useful theorem for calculating the center of mass is the parallel axis theorem which is stated as follows:

$$I_A = I_{cm} + Mh_A^2$$

where $h_A$ is the distance between the center of mass and the axis $A$, and $I_{cm}$ is the moment of inertia when the axis goes through the center of mass. The two axes (axis $A$ and the axis through the center of mass) must be parallel. (The proof of this is in section 11.7 of the book.)

- Given the theorem above, how can we see that the moment of inertia for any object is smallest through the center of mass?

- Was the moment of inertia smallest through the center of mass in problem one?

- Why is it reasonable (use your intuition) that the moment of inertia for any object is smallest through the center of mass?
4. **Moment of inertia for a hoop** - you have all the tools you need to solve these two problems below. Go back and see what equations seem useful.

- Find the moment of inertia of a hoop for an axis through its center if it has a mass $M$ and radius $R$. (Axis $A$ in the picture.) Recall that a hoop has essentially all of its mass at radius $R$.

- Find the moment of inertia of a hoop for an axis through its edge. (Axis $B$ in the picture.)
5. **Moment of inertia for a rod of uniform density** In this section we will find the moment of inertia of a rod of uniform density for an axis through its center. Take the rod to be of length $L$ and mass $M$. We will calculate the moment of inertia in the steps outlined below.

(a) Why will we have to use calculus here?

(b) Choose an origin to measure distances from. Mark it on the diagram. Be sure to measure all distances with respect to this origin.

(c) Break up the rod into lots of little chunks and find the mass of each chunk. Hint: what is the linear density (mass/length) of the whole rod?

(d) What is the contribution to $I$ for each chunk? Write total $I$ as a Riemann sum. Hint: you may find it helpful to distinguish between parameters (e.g. $M$ and $L$) that don’t change for a given object and variables (e.g. $x$ or $r$) that change even for a given object.

(e) Take the size of the chunk to zero so that the sum becomes an integral. What are the limits on the integral?

(f) Evaluate the integral.
(g) Does your answer for $I$ of the rod make sense? For example, should $I$ for the rod be more or less than if the mass were concentrated in two spheres a distance $L$ apart?

(h) Now use the same procedure to calculate $I$ for an axis through one end of the rod.

(i) Check your last two answers using the parallel axis theorem.
6. **Moment of inertia for a cylinder for an axis through its center** Take the total mass to be $M$, height to be $h$, and the radius to be $R$. Again here, you will have to use calculus.

- As you break the cylinder into chunks, remember that each chunk must have a single value of $r$ - the distance to the axis. How will you chunk the cylinder given this constraint?

- What is the density (mass per volume) of this cylinder?

- What is the mass in a chunk?

- What is the contribution to $I$ for each chunk?

- What is the Riemann sum for $I$? What is the corresponding integral and limits?

- Evaluate the integral.

- Note that we can’t do all shapes because it requires multi-dimensional calculus. The table on page 249 will help on homework and will be provided on tests if needed.
Part VI

Oscillations and Waves
Activity 11

Description of Motion for Object on a Spring

For the next few weeks we will be studying objects moving back and forth. Today we will focus on describing one of the simplest examples of this: an object hanging on a spring.

You should have at your lab bench a spring, a set of objects of different mass, stand to hang the spring on, a stopwatch and a ruler. Hang the spring on the stand and one of the objects on the spring.

- **Description of Motion with Plots:** We begin by describing the motion in plots.

  1. Pull the object down a bit from its resting position and then let the object go. Sketch \( x(t) \) for the motion on the plot provided on the next page. (Ignore the scales on the axis, just sketch the motion.)

  2. Using your your knowledge of calculus and your \( x(t) \) plot, sketch \( v(t) \), and \( a(t) \) for the object on the axes provided.

  3. Consider your plot of \( v(t) \). Does it agree with what you see? For example do both the plot and the actual object have zero velocity at the same time? Do they both either keep the same sign of velocity or both change signs?

  4. If physical reality and your plot do not agree, alter your plot so that they do agree.

  5. Is the function for \( x(t) \) most likely to be a polynomial, exponential, logarithmic or a trigonometric function of time?
ACTIVITY 11. DESCRIPTION OF MOTION FOR OBJECT ON A SPRING
• **Description of the motion with equations:** Consider motion that is described by a sine or a cosine function:

\[ x(t) = A \sin(\omega t + \phi) \]

where \( A, \omega \) and \( \phi \) are fixed for a given object hanging on a given spring, but may change if you change the properties of the object or the spring. Variables that are fixed for a given system but vary from system to system are called **parameters**.

In the following we will uncover the effect of the parameters on the behavior of the function.

1. We will begin by looking at \( x = \cos \theta \) and \( x = \sin \theta \), the simplest form of trigonometric functions. Sketch these two functions on the axes provided on the next page. Hint: if you forget which one begins at zero and which one begins at one, think of their definitions in terms of ratio of sides of a triangle.

   (a) What is the maximum value of each?

   (b) What is the minimum value of each?

   (c) How long does it take the function \( x = \cos \theta \) to repeat? That is, how long does it take for the function value and the derivative to return to the initial values?

2. Investigation of the parameter \( A \)

   (a) On the same axes as the cosine (with a different color pencil), sketch a plot of \( x = 3 \cos \theta \). What is the maximum value of this function?

   (b) On the same axes (with a different color pencil), sketch a plot of \( x = \frac{1}{3} \cos \theta \). What is the maximum value of this function?

   (c) State in words what the parameter \( A \) tells you about the function \( x(\theta) = A \cos(\theta) \)

   The technical term for this parameter \( A \) is amplitude. It is sometimes also written as \( x_{\text{max}} \).
3. Investigations of the the parameter $\omega$.

(a) On the same axis as the sine (with a different colored pencil), sketch a plot of $x(\theta) = \sin(2\theta)$. How long does it take this function to repeat?

(b) On the same axis as the sine (with a different colored pencil), sketch a plot of $x(\theta) = \sin(\theta/2)$. How long does it take this function to repeat?

(c) Based on your work with the previous two questions, how long will it take $x(\theta) = \sin(a\theta)$ to repeat (your answer will be in terms of $a$)? Hint: at what value of the argument does the function begin to repeat.

(d) So far, our independent variable has been $\theta$, but in describing motion, our independent variable is $t$, so that $x(t) = A\sin(\omega t)$ If $t$ is in units of seconds, what units must $\omega$ be in? Hint: what must be the units of the argument inside the sine or cosine?

(e) State in words what the parameter $\omega$ tells you about the function $x(t) = A\cos(\omega t)$.Hint: at what value of the argument does the function begin to repeat?

The parameter $\omega$ is often called the angular frequency. It is the same variable that we used in describing motion in a circle.

(f) What is the relationship between period and angular frequency?
4. Velocity and Acceleration

(a) Beginning with the general form of \( x(t) \) as a cosine with amplitude \( x_m \) and angular frequency \( \omega \), derive \( v(t) \) and \( a(t) \) for the same motion.

(b) What is the amplitude of the velocity? What is the period of the velocity?

(c) What is the amplitude of the acceleration? What is the period of the acceleration?
5. Investigations of phase

(a) Do sine and cosine differ in amplitude? period? Do they differ in any way? If so, describe that difference in non-technical language.

(b) Plot $x(t) = \cos(\theta + \pi/2)$.
(c) Describe in words how the $\pi/2$ changed the function.

(d) Plot $x(t) = \cos(\theta - \pi/2)$
(e) Describe in words how the $-\pi/2$ changed the function.

(f) State in words what the parameter $\phi$ tells you about the function $x(t) = A\cos(\omega t + \phi)$

The parameter $\phi$ is often called the phase shift.

(g) How would you initiate the motion of an object on a spring to give a cosine motion?

(h) How would you initiate the motion of an object on a spring to give a sine motion?
Activity 12
Forces for Object on a Spring

Last class we described the motion of an object hanging on a spring, today we will explain that motion in terms of forces acting on the object. You should have at your lab bench a spring, a set of objects of different mass, a stand to hang the spring on, a stopwatch and a ruler. Hang the spring on the stand and one of the objects on the spring.

- **Examining the forces:** We will consider for this a block of mass $m$ attached to a spring moving horizontally on a frictionless surface. (Hanging the spring vertically only changes the equilibrium position, everything else is the same. The vertical setup is much easier experimentally which is why we use that one.)

1. What forces are acting on the object if you displace it to the right of equilibrium? Draw a free body diagram.

![Free body diagram](image)

2. What is a reasonable choice for the location of $x = 0$?

3. What is the net force acting on the object at any arbitrary position (relative to your choice for $x = 0$)? Express your answer in terms of the parameters $k$ and the variable $x$. Be careful about the sign of the force: positive is to the right, negative is to the left.

4. Does this net force tend to move it closer to or further from equilibrium?

5. What is the acceleration of the object at any arbitrary position in terms of $k, m$ and $x$?
• **Equations of motion** Last class we began to **suspect** that the mass on a spring moves according to \( x(t) = x_m \cos(\omega t + \phi) \). Here we will **prove** that this is so by looking at the differential equation for the block on a spring. (You will use the Taylor Expansion method for solving the same differential equation in calc class soon.)

1. Copy down the expression for the acceleration of the block on the spring that you found on the first page.

2. Using the above equation, write down the expression for \( \ddot{x} \) for the object on a spring in terms of \( k, m \) and \( x \). Use that acceleration = \( a = \ddot{x} \). The equation that you get is known as **the differential equation for a block on a spring**.

3. Note that \( \ddot{x} \) is equal to a constant times \( x \). We will begin one step simpler: what function that we know has a **first** derivative proportional to the function itself? That is, what function solves \( \dot{y} = by \) where \( b \) is a constant?
4. Based on what we saw on the last page, a good guess for the solution is $x(t) = e^{bt}$. Now we will determine if our guess correct. If it is, what is the value of $b$? We will do this in several steps:

- Use the guess $x(t) = e^{bt}$ to evaluate both $x$ and $\ddot{x}$.

- Write down the differential equation from the last page that we are trying to solve.

- Plug the expressions for $x$ and $\ddot{x}$ from our guess in the differential equation.

- Now evaluate $b$ in terms of $m, k$ and $t$.

- We have a solution if $b$ is a constant, independent of time. Is your expression for $b$ independent of time?
5. On the last page we verified that we have a solution to the differential equation. On this page we will see, however, that it is not general enough to describe all possible motions and we will have to modify the solution a bit.

Imagine we have a block on a spring that has an initial amplitude $x(0) = 5$ cm. How can we modify the solution on the last page so that this initial condition is satisfied?

Does the new solution still solve the differential equation? Hint: use the new solution to give expressions for $x$ and $\ddot{x}$ and repeat the procedure on the last page.

Imagine again that our block has an initial velocity of $\dot{x}(0) = 0$ cm/s. Does your modified solution satisfy this initial condition?
You should have found that your new solution could not give an initial velocity of zero. The resolution to this difficulty is to note that $b$ has two roots, one positive and one negative. Show that if $x(t)$ is the sum of the two solutions ($x(t) = Ae^{bt} + Be^{-bt}$), it is still a solution to the differential equation.

Evaluate the constants $A$ and $B$ in $x(t) = Ae^{bt} + Be^{-bt}$ to give $x(0) = 5\text{cm}$ and $\dot{x}(0) = 0 \text{ cm/sec}$. 

In Search of Newton

Calculus and Physics
This page is for those who finish early:

6. If we have that \( x(0) = 0 \) cm and \( \dot{x}(0) = 2.3 \) cm/sec, what is the solution to the differential equation?

7. If we have that \( x(0) = 3.1 \) cm and \( \dot{x}(0) = 2.3 \) cm/sec, what is the solution to the differential equation?

8. According to your solution on the last page, what is \( \omega \) for your system? Does this agree with your experiments? Hint: what units does \( b \) have?
Activity 13

Fourier Series

In this activity we will investigate how periodic functions can be written as a series of sine and cosine terms (known as a Fourier series):

\[ f(t) = a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t) + a_2 \cos(\omega_2 t) + b_2 \sin(\omega_2 t) + \ldots \]

and begin to understand how to determine the Fourier coefficients \(a_1, b_1, \ldots\) given \(f(t)\).

1. **Finding the Coefficients and frequencies by eye** Off of the Isle of Shoals the Oceanography Department has a small buoy. One of the things that it measures is the depth of the water. A plot of the depth of the water over a two day period is given in the plot in the figure below. Use the figure to answer the questions below.
(a) What is the difference in the depths between low and high tide?

(b) How long does it take for a tidal cycle to complete? What is the frequency for the tide?

(c) What is the “average depth”?

(d) How high are the waves that the buoy rides on?

(e) What is the frequency of these waves?

(f) Find a formula that approximates the depth of the water over time.

You have just evaluated a Fourier series by eye.
2. In the last example you were able to find the frequencies and coefficients by eye. Below is a periodic function with a flat top. In this exercise you will see that even such a function as this can be represented as a sum of cosines. We will have to give you the coefficients and frequencies since these cannot even be estimated by eye.

(a) Plot each of the following functions from $t = 0$ to $t = 6\pi$:

(a) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t)$.

(b) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t) - \frac{8}{4\pi} \cos(2t)$.

(c) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t) - \frac{8}{4\pi} \cos(2t) - \frac{4}{9\pi} \cos(3t)$.

(d) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t) - \frac{8}{4\pi} \cos(2t) - \frac{4}{9\pi} \cos(3t) - \frac{4}{25\pi} \cos(5t)$.

(e) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t) - \frac{8}{4\pi} \cos(2t) - \frac{4}{9\pi} \cos(3t) - \frac{4}{25\pi} \cos(5t) - \frac{8}{36\pi} \cos(6t)$.

(f) $\frac{3\pi}{4} - \frac{4}{\pi} \cos(t) - \frac{8}{4\pi} \cos(2t) - \frac{4}{9\pi} \cos(3t) - \frac{4}{25\pi} \cos(5t) - \frac{8}{36\pi} \cos(6t) - \frac{4}{49\pi} \cos(7t)$.

How do these functions compare to the one plotted above? What is the next function in the sequence?

(g) Plot the function $\sum_{n=1}^{\infty} \frac{2}{n} \sin(nt)$. 
3. **Using Integration to evaluate coefficients** Suppose that you measure the voltage at a particular point in a circuit and that the voltage is combination of sines and cosines. How can you figure out what the correct combination is? Here we will look at a known signal, $1.7 \sin(t) + 0.85 \sin(2t)$ and will examine how integration can be used to isolate the coefficients, 1.7 and 0.85.

(a) Plot the voltage as a function of time from $t = 0$ to $t = 8\pi$. Draw a rough sketch below:

(b) Plot each of the following functions from $t = 0$ to $t = 2\pi$. Next to the function provide an estimate of what you think the integral of the function is by estimating the area under each plot.
   i. $\sin(2t) \sin(t)$.
   ii. $\sin(t) \sin(t)$.
   iii. $\sin(2t) \sin(2t)$.

(c) Use Riemann sums with 8 panels to approximate the following integrals:
   i. $\frac{1}{\pi} \int_{0}^{2\pi} (1.7 \sin(t) + 0.85 \sin(2t)) \sin(t)dt$.

   ii. $\frac{1}{\pi} \int_{0}^{2\pi} (1.7 \sin(t) + 0.85 \sin(2t)) \sin(2t)dt$. 
4. **Finding the coefficient on your own using integration.** Another measurement is made, and you are told that the new circuit should also be a sum of \( \sin(t) \) and \( \sin(2t) \). If the data collected is given in the following table, find a formula for the voltage. The time goes from 0 to \( 2\pi \).

<table>
<thead>
<tr>
<th>Time (milliseconds)</th>
<th>Voltage (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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Activity 14

Problem Solving: Damped motion

Instructions: This is the first of several problem solving days that we will have this semester. Here are the rules:

• Solve the problem using the GOAL format.

• Turn in one sheet at the end of the day with your group’s work. This will be graded on how well you follow the GOAL format and on the correctness and completeness of your solution.

• All groups must answer the first question below. The last two questions are supplementary for groups that finish the first question early.

It is your first week on your new job, helping to design magnetic braking systems. You are a bit anxious since you don’t know much about magnetism, but your boss assures you that your knowledge of oscillations is what she needs, and that you will know all about magnetism by the end of April.

She tells you that you can model (approximate) the magnetic damping force by \( bv \) where \( b \) is a constant and \( v \) is the velocity of the object. The damping force is always in the opposite direction to the velocity.

• As a beginning to the project, she says we want to figure out how to damp as quickly and cheaply as possible the motion of a metal block on a spring, moving on a horizontal surface with negligible friction. The spring has a constant of \( k = 5000 \) N/m and the mass of the block is .5 kg. She wants to know what is the smallest (and therefore cheapest) value of \( b \) will damp the amplitude of the motion to 10% of its original amplitude in .1 second.

• As a next step, she wants to know what will happen for all values of \( b \) and the given values of \( k \) and \( m \) - at least qualitatively. Do all values of \( b \) give oscillations, or do some values do something different?

• And finally, she want to know what types of general behavior occur for all values of \( b \) and \( k \) (\( m \) is still 500 g). Again, she want to know qualitative behavior, not details.
Part VII

Electrostatics
Activity 15

How fast is the comet going?

A comet is hurtling through space directly towards earth. The comet is first noticed traveling at 3000 m/s as it passes by the moon. It has a radius of 5 km. The citizens turn frantically to you – an expert in gravitation and electrostatics – to save the world.

As a first step in this important project, you model this as a one-dimensional problem with the comet on the line between the earth and moon, headed straight toward the earth. The comet is initially $2 \times 10^6$ m from the center of the moon. You then calculate with what speed the comet will hit the earth.
ACTIVITY 15. HOW FAST IS THE COMET GOING?
Activity 16

When will the comet hit?

We want to continue solving the problem from last week, concerning a comet traveling straight at the earth, with an initial speed of 3000 m/s next to the moon. The comet has a radius of 5 km. In this exercise we will find out how long it will take the comet to hit the earth $= t_{\text{hit}}$.

- We know from last week that the acceleration is not constant, and the details of the motion given by $x(t)$ are more complicated that we are used to. We begin by getting some estimates of time:

  1. We are going to use $a = (v_f - v_i)/t_{\text{hit}}$ to approximate the time. What is $v_f - v_i$ for this problem? (Use your numbers from last week.)
2. To get an estimate for the time, you will calculate the acceleration at three different positions. Your group will be assigned only one of the calculations below. Be sure to write your result on your white board for the rest of the class to see.

- What is the acceleration the comet feels initially? (Remember, to make our life easy we are ignoring the attraction of the comet to the moon.) Using that acceleration and the above equation, find \( t_{\text{hit}} \) in hours.

- What is the acceleration it feels at the surface of the earth? Using that acceleration and the above equation, find \( t_{\text{hit}} \) in hours.

- What is the acceleration it feels 2/3 of the way toward the earth? Using that acceleration and the above equation, find \( t_{\text{hit}} \) in hours.

3. Which of the above values is the best estimate for the actual time? Explain.

4. Make a qualitative sketch of \( x(t) \) for the comet as it moves toward the earth. Hint: is velocity larger at the beginning or at the end?
What we have just finished is one key to good scientific programming - you must have some ball park idea of the answer before you use the computer, otherwise you cannot write reasonable code or check its accuracy. You already have a good idea of the total time and the qualitative look of $x(t)$.

- Now you want to write a short Matlab code that will use Euler’s method to find $x(t)$.

1. What are the initial values of position and velocity for the comet? Don’t forget to setup a coordinate system, mark the origin, and decide on positive and negative directions for $x$.

2. What would be a good value for $\Delta t$? Hint: To pick a good value for $\Delta t$, $\Delta v$ must be small at each time step.

3. How do you calculate $v_i$ from $v_{i-1}$ and other variables using Euler’s method?

4. How do you calculate $x_i$ from $x_{i-1}$ and other variables using Euler’s method?

5. When should you make the program stop calculating?

6. As a check, are your signs on $x, v,$ and $a$ correct? Explain how you know.
ACTIVITY 16. WHEN WILL THE COMET HIT?

7. Using this information, write your matlab code. We have provided a code template (look for 408 student/calcphys/comet.template.m). Before making any changes save this file as "yourname." in the same folder. Then fill in the missing information.

8. How should total mechanical energy change as the comet moves? How can you use this as a check on your code?

- The last thing you need to do is run the code and make sense of the answers.

1. Run the code and look at the output.
2. Does \( x(t) \) look as you would expect? Sketch and explain.

3. How long did it take to hit the earth? Is this in reasonable agreement with your original estimate?

4. If you take \( \Delta t \) half as big, should your answer change a lot? Explain.
5. Take $\Delta t$ half as big and rerun the code. What is the time to hit now?

6. Based on the comparison of your answers from the two runs, is your code working well? Explain.

7. Where is Euler’s method least accurate for this problem? Explain.

8. Now imagine that you want to write your own version of electric field hockey. Could you use Euler’s method to do it? Could you do it a simpler way? What would be different from the comet problem? What would be the same?
Activity 17

\(\vec{E}\) field due to a bar of charge

Consider a thin bar with uniform charge density that has a total charge of 16\(\mu\)C and a total length of 24 cm. (Recall that \(k = 90 \text{ N cm}^2/\mu\text{C}^2\). We will calculate the \(\vec{E}\) field at a point \(P\), 9 cm directly to the left of the bar. (This point is said to be collinear with the bar because it is on the same line.) We will do this in several steps.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Figure 1.}
\end{figure}

1. Getting bounds on the answer First, we’ll get some rough bounds on the electric field by finding the force if all the charge were at one place. To save time, have each group will do only one calculation and write their results on the white board to share with others.

   (a) Charge concentrated at the center. What would be the electric field at \(P\) if all of the charge were concentrated at the center of the bar?
   
   direction:
   
   magnitude:

   (b) Charge concentrated at left. What would be the electric field at \(P\) if all of the charge were concentrated at the left of the bar?
   
   direction:
   
   magnitude:

   (c) Charge concentrated at right. What would be the electric field at \(P\) if all of the charge were concentrated at the right of the bar?
   
   direction:
   
   magnitude:

   (d) Compare your answers. Which value should be largest? smallest? Are they?
2. Deciding on an approach - looking at the general case

Let’s forget our particular case of finding the electric field due to a bar of charge, just for a moment. The general formulas for finding the electric field due to a single point charge at a point \( P \) is

\[
E_x = \frac{kq_1 \cos \theta_1}{r_1^2} \quad E_y = \frac{kq_1 \sin \theta_1}{r_1^2}
\]

where the variable correspond to the those in the sketch.

(a) What is the formula you would use to find the \( E_x \) due to 2 point charges?

(b) What is the formula you would use to find the \( E_x \) due to \( N \) point charges?

(c) How would you go about finding \( E_x \) and \( E_y \) for a bar of charge? (Don’t do the calculation, just describe the general procedure you would use.)

3. Deciding on an approach - our specific case

(a) Now we return to consider our particular case of finding the electric field due to a bar of charge at a point just to the left of the bar (see Figure 1). Looking back at your formula for the general case of \( N \) charges, which terms can change depending on which charge we are considering, and which terms are constant? In particular, how do the cos and sin terms vary in this case?

(b) Edit your general formulas for \( E_x \) and \( E_y \) by putting an \( i \) subscript on each term that varies from charge to charge. Since the angle does not change, go ahead and evaluate the sine and cosine terms and put those exact values in the sum.
4. **Setting up the integral** You probably suspect by now that we have a Reimann sum and we are headed toward an integral. And you are right.

(a) State in words what physical entities we are summing over.

(b) In order to have a sum go to an integral, the size of the chunks have to get smaller and smaller. What term in the sum is getting smaller and smaller as we consider smaller chunks?

(c) Rewrite the sum by putting a \( \Delta \) in front of the quantity that gets smaller and smaller.

(d) Rewrite the sum as an integral (in the limit that the chunks get very small).

(e) Does this look like an integral you know how to do? If not, what is disturbing about the integral? What is causing you confusion? (We’re expecting that this is not a friendly looking integral yet!!)
5. Evaluating the integral - step 1 Now we will try to deal with the troublesome parts slowly. First we will deal with the $r^2$ term, then with the $dq$ term. The $k$ is a constant and should cause no real trouble.

(a) The key to making sense of $r^2$ is putting down a coordinate system. In the sum we label the terms in the sum by $i$. When we go to an integral we want to label terms by their location $x$. To do this we need a coordinate system and an origin (so we know what $x$ means). Sketch the bar and a coordinate system below and the point $P$ as shown in Figure 1. Make any reasonable choice for the coordinate system, and we can see later if another choice would have been better.

(b) Consider a chunk of the bar located at position $x$. What is $r$ for that chunk of the bar?

(c) Use this information to rewrite the integral with $r$ in terms of $x$.

6. Evaluating the integral - part 2 Now we will deal with $dq$.

(a) Why is the $dq$ term confusing (if it is)?

(b) In general, we would expect a $dx$ term. It is not okay to replace $dq$ with $dx$. Why not?

(c) It is not okay just to put a $qdx$ in the integral in place of $dq$. Why not?
(d) What we have to do is find an exact relationship between $dq$ and $dx$. Let’s begin by clarifying what these terms are. In words, $dq$ is the charge in a little chunk of the bar, $dx$ is the length of a little chunk of the bar. Given the data on the front sheet, calculate the following:

i. If $\Delta q = 16 \mu C$, what is the corresponding $\Delta x$? (We’ve switched to $\Delta$ since the chunks aren’t small here).

ii. If $\Delta q = 8 \mu C$, what is the corresponding $\Delta x$?

iii. If $\Delta q = 4 \mu C$, what is the corresponding $\Delta x$?

iv. Hopefully, you’ve noticed a pattern, that $\Delta q$ is proportional to $\Delta x$. Write down the proportionality relationship. The proportionality constant must have units! Use units to help you check your relationship.

v. Explain this relationship in words.

7. Use the relationship you just developed to replace $dq$ in the integral.

8. What are the limits on the integral?
9. Evaluating the integral - part 3

(a) You should now have an integral that looks familiar. Go ahead and integrate.

(b) As a first check, is your answer between the bounds that we calculated on the first page?

(c) Is the value the same as if all the charge were at the center? Should it be?

(d) As a second check, are the units correct?

(e) The best coordinate system puts the origin at point $P$. Do you see why this is best? Hint: what does the integral look like if you put the origin at the left end of the bar?

(f) Why did we state that the bar was thin? How would the problem change if we had a bar that was thick? (Don’t solve the problem - just see why it is too hard for us!)
10. Extensions.

(a) Redo the same problem as above, but use \( d \) for the distance 9 cm, \( L \) for the length of the rod and \( Q \) for the charge on the rod. You can look at your answer and check that in the limit that when \( d \) is much much greater than \( L \), the electric field is that for a point charge.

(b) Do a slightly more difficult integral, finding the electric field for a point above the bar. The biggest change here is that the sine and cosine terms change from chunk to chunk. If you want to use numbers, go ahead and substitute numbers for \( d, L \) and \( q \). If you don't know how to start, go back and redo all of the steps that we did for the first problem. You may find that you have an integral you do not yet know how to do. Have no fear - you will learn how to deal with it in calculus this week!
ACTIVITY 17. $\vec{E}$ FIELD DUE TO A BAR OF CHARGE
Activity 18

Calculating $V$ due to a Charge Distribution

1. Find the electric potential due to a bar of uniform charge at point $P$ a distance $d$ from the left end. Begin with the general formula used for finding $V$ due to a charge distribution.

$$V = \int \frac{k dq}{r}$$

![Diagram of a bar charge distribution with points P and Q and distance d]
2. Is your answer on the last page in the correct units?

3. Is your answer on the last page correct in the limit that \( d \) is much much greater than \( L \)? That is, do you get the electric field for a point charge in that limit? Hint: use the Taylor expansion for the natural log: \( \ln(1 + x) \approx x \) for \( x \) much less than one.

4. First an aside: when you did the integral on the last page, \( d \) was a parameter, and \( x \) indicated the location of the charge on the bar. Now that the integral over the bar is done, we will let \( x \) take the place of \( d \) (because we want it to change) and look at \( V(x) \). Now the question: Is \( V(x = \infty) = 0 \)? Why should it be?

5. Take \( k = 1, q = 1, L = 1 \) and plot \( V(x) \). Sketch it below. Does it look reasonable? Explain.

6. Given that \( E_x(x) = kq/x(x + L) \), and that \( E_x(x) = dV/dx \), take the derivative of the potential and see if it gives you the correct \( E_x \).
7. Consider a point particle of mass 1 kg and charge $10 \mu C$ that we release from rest an infinite distance to the left of the bar. Using your results for the bar, and letting $Q = 16 \mu C$, $L = 24$ cm, find the velocity of a particle of mass as a function of distance away from the left edge of the bar.

8. Repeat the same calculation but for a particle of charge $20 \mu C$ and the same mass.
ACTIVITY 18. CALCULATING V DUE TO A CHARGE DISTRIBUTION
Activity 19

Neon Lights

There are many types of fluorescent lights. These lights are essentially just gas sealed in a glass tube. The gas, when heated, gives off light. Neon, for instance, emits red colored light. Since the color of the light is different for different gases, the color must somehow depend on the structure of the atoms in the gas.

In the 1920’s Neils Bohr developed a theory that stated that atoms are composed of electrons moving in uniform circular motion about a heavy positively charged nucleus. One of the key features of the theory was that only certain orbits were allowed - ones that had certain values of angular momentum. Angular momentum, Bohr postulated, was quantized, meaning that it could only have certain discrete values. The allowed values of angular momentum are integer multiples of a constant called Planck’s constant, $L = nh/2\pi$, where $L$ is the angular momentum, $h$ is Planck’s constant, and $n$ is an integer.

When an electron makes a transition from a high energy level to a lower energy level, light (a photon) must be emitted in order to conserve energy. The energy of a photon is related to its frequency by the relationship $E = h\nu$ where $h$ is Planck’s constant and $\nu$ is the photon frequency.

You want to test this theory by calculating the color of light emitted by the simplest atomic system, the hydrogen atom, and comparing it to the color you observe.

Use your books to find the information you need to solve this problem. Before you start your calculations, write down a detailed strategy for how you will approach the problem and have it checked by one of the instructors.

The following information may also be useful:

- Planck’s constant is $6.63 \times 10^{-34}$ Joule sec

- The radius of the smallest allowed electron orbit in hydrogen is $.529 \times 10^{-10}$ meters. The second allowed orbit has a radius 4 times larger.
Part VIII

Circuits
Activity 20

Discharging an RC Circuit

So far we have considered circuits with just capacitors or just resistors, now we consider a simple circuit with a switch, a capacitor, and a resistor, all in series. This is known as an RC circuit (R for resistor, C for capacitor). This is a long worksheet - we will solve the problem in the following pieces

- Understand the behavior qualitatively
- Write down the differential equation
- Understand the solutions to the differential equation qualitatively
- Solve the differential equation
- Understand the solution and connect it to our intuitions.
• Understand the circuit qualitatively Imagine that the capacitor has been charged up to a charge $Q_0$, and the switch is then closed. (In all of the following feel free to use your analogies with water flow or traffic flow.)

1. Does current flow through the resistor after the switch is closed? Explain. If it does flow, show the direction that the “positive charge” (recall that it is really electrons that move, but “conventional current” has the protons moving) move through the circuit.

2. Does the current flow forever, or does it stop? Explain. Use your intuitions about the forces on charged particles.

3. Will the capacitor discharge faster or slower if I increase the resistance but keep all else the same? Explain.

4. Will the capacitor discharge faster or slower if I increase the capacitance and keep the initial charge the same? Explain. Hint: use the relationship $Q = CV$ to answer this question.
• The differential equation

1. Use Kirchhoff’s loop rule (that is, the potential change around any loop is zero) to write the equation which describes the circuit. Your answer should have both a $q_c$ (the charge on the capacitor) and an $I_r$ (current through the resistor), as well as the parameters $R$ and $C$.

2. $I_r \equiv dq_r/dt$ - it is the rate of flow of charge through the resistor. Since the charge on the capacitor is decreasing and the current is positive, it must be true that $I_r = -dq_c/dt$. Make sure that this statement makes sense to everyone in your group.

3. Put together the two previous equations to find an equation for $dq_c/dt$ in terms of $q_c$ and the parameters of the problem. This is our differential equation for the charge on the capacitor.

• Understanding the solutions to the differential equation qualitatively Use the differential equation to answer the following questions:

1. What sign is the slope of the $q_c(t)$ curve?

2. Does the slope of the curve $q_c(t)$ increase or decrease as the capacitor discharges?

3. What is the initial charge on the capacitor?

4. What is the charge on the capacitor a long time after the switch is closed (i.e., what is the steady state)?
Below is the slope field for this differential equation.

5. Verify that this slope field corresponds to your differential equation by checking that the sign and relative magnitude of the slope in the plot agree with the differential equation.

6. Sketch the solution for an initial positive charge of 0.15 C.
7. Sketch the solution for an initial negative charge of −0.05 C.
8. From the slope field, what is the steady state solution and does this agree with your intuitions?
• Solve the differential equation analytically to find $q_c(t)$.

• Use the initial value $q_c(0) = Q_0$ to evaluate the constant.

• Understand the solution and connect it back to our qualitative predictions

  1. In your equation, you should find that $RC$ appear together. Define a new variable $\tau \equiv RC$ and rewrite the equation in terms of $\tau$. (On the next page we will find out what we can say about a circuit if we know $\tau$.) $\tau$ (pronounced “tau”) is known as the **time constant** for this circuit.
2. Show that $\tau$ has units of time. Hint: write $R$ and $C$ in terms of voltage, current, and charge.

3. At a time $t = \tau$, what is the ratio $q_c(t)/Q_0$?

4. At a time $t = 2\tau$, what is the ratio $q_c(t)/Q_0$?

5. At a time $t = 3\tau$, what is the ratio $q_c(t)/Q_0$?

6. Use your last few answers to draw a slightly more accurate sketch of $q_c(t)$ on the grid below $\tau$.

7. After how long is $q = Q_0/10$? is $q = 0$? Express your answers in terms of $\tau$. Hint: combine the equation $q = Q_0/10$ with the equation for $q(t)$ to find and then solve an equation for time in terms of $\tau$.

8. Does the capacitor discharge faster or slower as $R$ and $C$ increase? Does this agree with your original predictions on the first page?
Activity 21

Charging an RC Circuit

We have just learned how a capacitor discharges through a resistor. Now we will examine how it charges up with a battery and a resistor if the initial charge is zero. We will go through the same steps, but more quickly this time.

- **Understanding the circuit qualitatively**
  
  1. If the capacitor is initially uncharged, will current flow immediately after the circuit is closed? Explain.

  2. Does the current flow forever, or does it stop? Explain.

  3. Will the capacitor charge faster or slower if the resistance is increase?
• **Differential equation** Now consider the same circuit, but with a battery. Such a circuit would be used to charge up the capacitor. The capacitor is originally uncharged.

1. Use Kirchhoff’s loop rule to write the equation which describes the circuit. Your answer should have both a $q_c$ and an $I_r$ and parameters.

2. Note that since both $I_r$ and $dq_c/dt$ are positive this time, then we have $I_r = dq_c/dt$. Make sure that everyone in your group is comfortable with this statement.

3. Use your answers in the previous questions in order to give an equation for $dq_c/dt$ in terms of $q_c$ and the parameters of the problem. This is the differential equation for the problem.
• **Investigate the solution qualitatively** Based on the differential equation, answer the following questions:

1. What is the steady state charge on the capacitor? That is, use the differential equation to find what is the charge when the derivative of the charge is zero. (We will call this $q_{ss}$.

2. What sign is the slope of the $q_c(t)$ curve if $q > q_{ss}$? if $q < q_{ss}$?

3. Does the slope of the curve $q_c(t)$ increase or decrease as the capacitor approaches $q_{ss}$?
Below is the slope field for this differential equation.

4. Given the data at the top of the plot, what is $q_{ss}$ for this system? Does this agree with the slope field?

5. Verify that this slope field corresponds to your differential equation by checking that the sign and relative magnitude of the slope in the plot agree with the differential equation.

6. Sketch the solution for an initial charge of 0 C.

7. Sketch the solution for an initial charge of .15 C.
• **Solve the differential equation** to find $q_c(t)$, being sure to use the initial conditions.
• Understand the solution and connect it back to our qualitative predictions

1. Is \( \tau = RC \) still the time constant in this problem as it was for discharging the capacitor? Does this agree with your intuition? That is, should it take longer to charge the capacitor if \( R \) or \( C \) are increased?

2. After how long is \( q = 95\% \) of its steady state value if it is initially uncharged? Express your answer in terms of \( \tau \).

3. When does the charge on the capacitor reach its steady state value?

4. RC circuits are characterized by their value of \( \tau \). Why is this reasonable?

5. Describe what happens if the initial charge on the capacitor is greater than \( q_{ss} \). Explain why this should be so.


Activity 22

Design of an RC Circuit

In this activity you will design a circuit that meets the following design specifications:

1. The power supply voltage is set at 4 Volts

2. The capacitor charges to 90% of its steady state value in as close to 5 seconds as possible with the given equipment. You must use the one resistor and capacitor given to you, plus one more element (either a resistor or capacitor) located on the front table.

• Design the Circuit - Stage One On a separate piece of paper (to be handed in) do the calculations necessary to predict \( V(t) \) for the voltage across the capacitor with your given resistor and capacitor.

• Setting up

1. Turn on the Mac and open Electricity.

2. Put one graph on the screen to plot voltage. Fix the time scale and the voltage scales based on what you know about the circuit you need to design.

3. Fix the settings under the Collect Menu:
   (a) All Graphs Live (the graphs will be plotted as you take data)
   (b) Display Inputs (the numerical value of the voltage will be displayed on the bottom of the screen).
   (c) Select Inputs: Port1 only
   (d) Data Rate: 20 points per second
   (e) Averaging: 15 point averaging for Port1
   (f) Triggering: Port1 > 0

4. Calibrate the voltages:
   (a) To calibrate the voltage you will need to measure the voltage directly across the power supply as shown. You will also need to hook up the voltmeter across the power supply to give the calibration voltage. Here are the details of the setup:
i. The buffer amplifier is the small homemade metal box with a switch. The connections on the “in” side go to the battery, the connections on the “out” side go to the dual channel amplifier. Turn on this amp before turning on the power supply.

ii. For the little voltmeter, be sure you have selected DC volts. For the voltmeter, one banana plug is placed in the “com” position, the other is placed in the “V/Ω” position. Turn the meter on.

iii. The Dual Channel amplifier is marked with its name. Be sure to use the “probe” pin connected to the banana plugs, not the ones connected to small boxes labeled “current”.

(b) Under Collect choose Calibrate, then Calibrate Now. You need to take readings at two voltages; choose about 1V and 5V. Follow the instructions on the screen.

c) Check that the calibration is correct by comparing the voltmeter and computer readings for several values between 1 and 5 volts.

d) Hook up the circuit as shown. You may also keep the voltmeter attached to give a reading of the voltage across the battery.

(e) Discharge the capacitor by running a short wire across the capacitor for a second or so. Verify that the voltage across the capacitor is zero (or nearly so) by looking at the computer reading.

• Taking Data

1. Set the voltage between 4 and 4.5 Volts; keep it there for the remainder of the session.
2. Press start on the computer, wait about ten seconds to allow the ULI to turn on, then quickly and firmly close the switch.

3. Print out a plot of the data.

- **Fit the data** Fit the data to see if it follows the expected functional form.

  1. Choose Analyze Data A from the Analyze menu. If a region of your data is bad (i.e., you started taking data before the switch was shut), be sure to highlight the good portion of your data before continuing.

  2. If only some of your data is “good”, select that section of the data.

  3. Choose Fit... from the same menu.

  4. On the left hand side of the Fit box, choose the functional form that you expect to see.

  5. Then choose Try Fit. Once the results are acceptable, choose Maintain Fit.

  6. Print the plot if possible, otherwise, write down the fit here:

  7. What is the value of $\tau$ from the fit?

  8. What is the maximum voltage from the plot itself?

  9. Are these in reasonable agreement with expectations?
• **Design and test your circuit** On another sheet of paper, do the calculations necessary to design your circuit to meet design specifications. Test your circuit to see if it meets design specifications. Fit the data and print out the plot.

• **If you’re done ahead of time...** Once you have come up with the best design possible, use your given resistor and capacitor, plus one extra resistor or capacitor, to give the longest time to charge possible, and then the shortest time.

• **Please be sure that the buffer amplifier is turned off when you go (along with everything else!)** Please be sure to return the extra resistors and capacitors to the front of the room and put them in their proper boxes!
Part IX

Magnetism
Activity 23

A new circuit element

You work for a small electronics firm. Your boss returns from a trade show, excited about a new circuit element called an inductor. He doesn’t know the details, and only knows that the voltage drop across the inductor is given by

\[ V_L = L \frac{di}{dt}, \]

where \( L \) is a constant value for an inductor (like the capacitance is constant for a capacitor) and \( i \) is the current. He also knows that the symbol used looks like a solonoid or a spring.

He asks you to figure out what happens if you put this inductor in a circuit with a charged capacitor (with charge \( Q_0 \)) but no battery. He wants to know the time behavior both qualitatively and quantitatively (with an equation) and if energy is dissipated or if it is conserved.

Once you have that system well in hand, he asks what happens (i.e. what is the time evolution) if you add a resistor to the above circuit. Is energy conserved in this case or not?
Activity 24

Martian Gas Gauge
Or how I came to love series and differential equations

You have been hired by marooned Martians to repair their broken-down flying saucer. Upon inspection, you are surprised to find that the saucer is powered by a nuclear reactor. The reactor works as follows: fuel rods (uranium) are immersed in heavy water (D₂O rather than H₂O) in order to moderate the fission reaction. Too much water stops the reaction completely, while too little water well, you know. It turns out that the Martians have no way to measure the water level in their reactors. You come up with an ingenious plan: you design a parallel plate capacitor that hangs from the same rack as the fuel rods. As the fuel rods are immersed, heavy water fills the space between the capacitor plates. Since water is a dielectric (k=75), the capacitance changes.

The capacitor is connected to a circuit as shown in the figure. An electronic timer changes the switch position back and forth so that the capacitor is connected to the battery (and therefore charging) half of the time, and disconnected (discharging) half of the time. The rate at which the switch operates can be adjusted. Just before the switch disconnects the battery from the capacitor, the voltage across the resistor is read and displayed on a blinking LED panel (the Martians love that kind of stuff!) This voltage will be proportional to the level of water in the reactor.

![Circuit Diagram](image)

1. Write down the differential equation describing the charge on the capacitor plates as a function of time when the capacitor is connected to the battery. If you are
truly virtuous you will apply Kirchoff’s law to the circuit. If not, you can look up the answer from the book or your notes.

2. Write down the most general form of the solution possible, leaving undetermined parameters where appropriate.

3. Assuming that the plates are initially uncharged, write down the particular solution - i.e. replace the undetermined parameters with values depending only on C, R, t, and V.

4. Does your solution make any assumptions about the charge on the capacitor at large values of time? Explain.

5. You adjust the electronic timer on your circuit so that the switch position changes every millisecond. Assume that your capacitor has a value of 0.5 nanofarads, the resistor has a value of 1 megaohm, and the battery is 12 volts. Calculate the charge on the plates at the end of the first charging interval.

6. How does the charge on the capacitor compare to the maximum possible charge on the plates?

7. Write down the differential equation describing the charge on the capacitor as a function of time when the switch is not connected to the battery. The same rules apply as for question 1.

8. Write down the most general solution to this differential equation, leaving it in terms of the appropriate undetermined coefficients.

9. Write the particular solution given that the initial charge on the capacitor is $Q_0$. What assumptions did you make to derive your solution?

10. The switch now changes positions and the capacitor starts to discharge. Calculate how much charge will be left on the plates at the end of the first discharge interval. Use your answer from question 5 for $Q_0$.

11. The switch changes back to the charging position. Is your solution from question 3 for the charge as a function of time now valid for the second charging interval? Explain.

12. How can you adjust this particular solution so that it satisfies the new "initial" conditions? Hint: start from the general solution (question 2) and apply your new initial condition.

13. Check your new solution to see if it satisfies the differential equation.

14. Your new solution for the charging interval should now depend on the amount of charge left from the previous discharge interval. Have an instructor check your results before you continue.
15. Instead of calculating the charge at the end of each interval numerically, combine your results from questions 9 and 12 to give an algebraic expression for the amount of charge on the capacitor after the second charging interval.

16. Use these results to find an algebraic expression for the charge on the capacitor at the end of the second discharge interval.

17. Repeat steps 15 and 16 until you recognize a pattern to the results. Write a general expression for the charge on the capacitor plate at the end of the nth charging interval. What form is your expression in?

18. For the circuit above, calculate the RC time constant. Compare it to the switching time interval. Discuss how this exercise would have changed if RC were very small or very large compared to the switching time. Would it have made sense to design your gas gauge with RC much larger or much smaller than the switching time constant?

19. Design a system that will read 1 volt if the gas tank is empty and 5 volts if the gas tank is full. The battery has a fixed voltage of 12 volts. The remaining parameters that can be adjusted are the switching time interval, the resistance, the plate size, and the plate separation. Outline your approach before you start.

20. What will you charge the Martians for your services?